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UNCLASSIFIED MRC-TSR-2886 DAAG29-80-C-0041 F/G 20/4 NL

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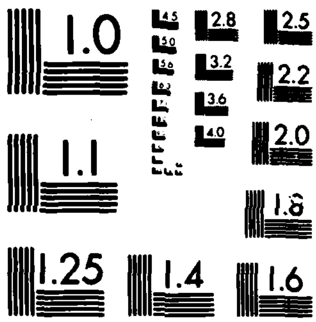

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MRC Technical Summary Report #2886

SOME REMARKS ON THE NAVIER-STOKES EQUATIONS WITH A PRESSURE-DEPENDENT VISCOSITY

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November 1985

(Received October 25, 1985)

Approved for public release
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Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

UNIVERSITY OF WISCONSIN - MADISON
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Abstract

We discuss the Navier-Stokes equations for an incompressible fluid with a viscosity that is allowed to depend on the pressure. Ellipticity and the complementing condition of Agmon, Douglis and Nirenberg [1] are discussed. It is found that the pressure dependence of viscosity leads to the possibility of a change of type. It is shown that the Dirichlet initial-boundary value problem is well posed as long as the equations do not change type.

AMS (MOS) Subject Classifications: 35J25, 35J55, 35J65, 35Q10, 76D05

Key Words: Navier-Stokes equations, pressure dependent viscosity, nonlinear Neumann problems, complementing condition, initial value problems, change of type.

Work Unit Number 1 (Applied Analysis)

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and supported by the National Science Foundation under Grant Nos. MCS-8215064 and DMS-8451761.

Significance and Explanation

In most mathematical treatments of the Navier-Stokes equations, it is assumed that the viscosity is a constant. Viscosities of real fluids, however, depend not only on the temperature, but may also change significantly with pressure, in particular at high pressures. In this paper, the mathematical consequences of such a pressure dependence are investigated. It is found that, in contrast to the ordinary Navier-Stokes equations, ellipticity can be lost, and the equations are not necessarily well-posed. The complementing condition for traction boundary conditions is investigated, and an existence theorem for the initial-boundary value problem with prescribed velocities on the wall is proved. One of the main differences to ordinary Navier-Stokes theory lies in the elimination of the pressure, which now leads to a nonlinear elliptic partial differential equation instead of Laplace's equation.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

SOME REMARKS ON THE NAVIER-STOKES EQUATIONS WITH A PRESSURE-DEPENDENT VISCOSITY

Michael Renardy

1. Introduction

In most mathematical treatments of the Navier-Stokes equations, it is assumed that the viscosity of the fluid is constant. Viscosities of real fluids, however, depend not only on the temperature, but may also depend on the pressure. In fact, the possibility of a pressure-dependent viscosity, even for incompressible fluids, is already mentioned by Stokes [8]. Stokes does not pursue the matter further, since experiments at his time did not suggest that this pressure dependence was important. More recent experiments at very high pressures, however, have shown considerable changes in viscosity. In [3], for example, various organic liquids were studied at pressures of up to several thousand atmospheres. It was found that for some of these liquids the viscosity increased by two orders of magnitude compared to its value at atmospheric pressure, while compression was only of the order of 10% [2]. Such data suggest that it is reasonable to consider a pressure dependence of the viscosity, even in fluids that can, to a good approximation, be regarded as incompressible. The results of [3] indicate that the dependence of viscosity on pressure becomes stronger as the pressure increases and that it is stronger in fluids that already have a high viscosity at atmospheric pressure. We note that the viscosities of polymer melts are several orders of magnitude higher than that of any of the fluids studied in [3].

From a theoretical point of view, we can think of incompressible fluids as a limiting case of compressible fluids. For a Newtonian compressible fluid, the stress tensor is given by

$$\mathbf{T} = -\phi(\rho)\mathbf{1} + 2\eta(\rho)(\mathbf{D} - \frac{1}{3}\operatorname{div} u\, \mathbf{1}) + \varsigma(\rho)\operatorname{div} u\, \mathbf{1}. \tag{1}$$

Here $\mathbf{D}$ denotes the symmetric part of the velocity gradient, and $\mathbf{D} - \frac{1}{3}\operatorname{div} u\, \mathbf{1}$ is the deviatoric (traceless) part of $\mathbf{D}$. The total "pressure" is given by

$$p = \phi(\rho) - \varsigma(\rho)\operatorname{div} u. \tag{2}$$

Noting that $\operatorname{div} u = -\frac{d}{dt}\ln\rho$, we see that (2) is a first order differential equation relating ρ to p. The solution of (2) is given by a functional expressing ρ in terms of the history of p. The incompressible limit arises by considering functions ϕ_n and ς_n such that, for any given p, the solution of (2) tends to a constant $\rho = \rho_0$ as $n \to \infty$. For example, we may

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take $\phi_n(\rho) = C_n(\rho - \rho_0)/\rho$, ς_n equal to a positive constant, and let C_n tend to infinity. If $\eta(\rho)$ is independent of n, then the incompressible limit yields the classical Navier-Stokes equation with constant viscosity $\eta = \eta(\rho_0)$. We may, however, consider a limiting case where not only $d\phi_n/d\rho$ becomes infinite as $n \to \infty$, but $d\eta_n/d\rho$, and perhaps ς_n, also become infinite. We must then in general regard η as a functional of the history of p, even in the incompressible case. In this paper, we restrict attention to the case where η is just a function of the instantaneous value of p. Of course we shall assume throughout the paper that the function $\eta(p)$ is always positive.

For the classical Navier-Stokes equations, it is well known that the stationary problem is elliptic in the sense of Agmon, Douglis and Nirenberg [1], that the time-dependent problem is "parabolic" and that both Dirichlet and traction conditions on the boundary satisfy the appropriate complementing conditions. This is of fundamental importance for studying the regularity of solutions to the stationary problem as well as for studying the time-dependent initial-boundary value problem. If the viscosity is dependent on the pressure, ellipticity can fail. This, and the complementing condition for Dirichlet and traction boundary conditions will be discussed in section 2.

If ellipticity fails, we can not expect well-posedness of the initial-boundary value problem. In section 3, we will, however, show a local existence result for the Dirichlet initial-boundary value problem under the assumption that the initial data are such that ellipticity is satisfied. The main complication compared to ordinary Navier-Stokes theory is that we can not use the usual Hodge projection to eliminate the pressure. Instead, we must solve a nonlinear elliptic equation for eliminating the pressure. The linearization of this equation at the current velocity and pressure defines a projection operator that depends on the velocity field and the pressure and assumes the role of the Hodge projection. This makes the theory considerably more complicated than that of the ordinary Navier-Stokes equations. Another difference is that we can of course no longer add a constant to the pressure without altering the flow. However, this indeterminacy in the classical Navier-Stokes theory has an analogue in the general case, and in order to obtain a unique solution, we must for example prescribe the average pressure as a function of time, in addition to the initial and boundary conditions for the velocity field.

2. Ellipticity and the Complementing Condition

The Navier-Stokes equations have the form

$$\rho(\dot{u} + (u \cdot \nabla)u) = \operatorname{div}\, [\eta(p)(\nabla u + (\nabla u)^T)] - \nabla p + f, \tag{3}$$

$$\operatorname{div}\, u = 0.$$

We can rewrite this in the form

$$\rho(\dot{u} + (u \cdot \nabla)u) = \eta(p)\Delta u + \{\eta'(p)[\nabla u + (\nabla u)^T] - \mathbf{1}\}\nabla p + f, \tag{4}$$

$$\operatorname{div}\, u = 0.$$

The ellipticity and complementing conditions concern the terms of highest differential order in the equation, and the coefficients of these terms are treated as though they were constant. That is, in order to discuss these conditions, we have to look at the problem

$$\rho \dot{u} = \eta \Delta u + (\mathbf{A} - \mathbf{1})\nabla p, \qquad (5)$$

$$\operatorname{div} u = 0.$$

Here the constant matrix $\mathbf{A}$ replaces $\eta'(p)[\nabla u + (\nabla u)^T]$, and it is therefore symmetric and traceless. For stationary problems we have $\dot{u} = 0$. The stationary Navier-Stokes system is elliptic if equation (5) (with $\dot{u} = 0$) has no non-constant periodic solutions on all of space.

It is not difficult to see that this is the case if and only if $\mathbf{A} - \mathbf{1}$ is negative definite (for the ordinary Navier-Stokes equations, $\mathbf{A} = \mathbf{0}$, and the equations are always elliptic). To see this, let us first assume that $\mathbf{A} - \mathbf{1}$ is negative definite. We multiply the first equation of (5) by $(\mathbf{1} - \mathbf{A})^{-1}u$ and integrate over one period. After integrating by parts, this yields

$$\int \sum_{i,j,k} (\partial_j u_i)(\mathbf{1} - \mathbf{A})^{-1}_{ik}(\partial_j u_k)\, dx = 0, \qquad (6)$$

which immediately yields $\nabla u = 0$. For the time-dependent problem, we obtain in the same fashion that

$$\frac{1}{2}\rho \frac{d}{dt} \int \sum_{i,j} u_i(\mathbf{1} - \mathbf{A})^{-1}_{ij} u_j\, dx = -\int \sum_{i,j,k} \eta(\partial_j u_i)(\mathbf{1} - \mathbf{A})^{-1}_{ik}(\partial_j u_k)\, dx, \qquad (7)$$

and hence periodic solutions with non-zero wave numbers decay exponentially with a rate proportional to the square of the wave vector, i.e. the equation is "parabolic".

If $\mathbf{A} - \mathbf{1}$ is not definite, the behavior changes. By taking the divergence in the first equation of (5), we see that the *equation for p is given* by

$$\operatorname{div}\,[(\mathbf{A} - \mathbf{1})\nabla p] = 0, \qquad (8)$$

and there are now nonconstant spatially periodic solutions to this equation. As a consequence, we can find nonconstant periodic solutions to (5) in the stationary case, and exponentially growing solutions in the time-dependent case. Solutions to the initial value problem are not unique, and if inhomogeneous terms are added to the equation, they may not exist. We conclude that, in contrast to the ordinary Navier-Stokes equations, the initial value problem is not always well-posed. Only as long as the eigenvalues of the symmetric part of the velocity gradient are less than $\frac{1}{2\eta'(p)}$ can we expect to prove a local existence result. If this condition is violated, problems of nonexistence and nonuniqueness occur in the constant coefficient problem and hence can be expected in the full Navier-Stokes system as well.

Roughly speaking, ellipticity is a condition concerning the local behavior of solutions near points away from the boundary of the domain in which the equations are studied. Near the boundary, however, the nature of the boundary conditions is important, and "well-behaved" problems are characterized by a restriction on the boundary conditions, in addition to ellipticity. This extra condition is known as the complementing condition. The boundary is regarded as locally flat, and one looks at a half-space problem. Again we consider the terms of highest differential order in the equations, and we look for solutions of the half-space problem that are nonconstant and periodic in the directions parallel to the boundary of the half space and decay exponentially away from the boundary. The complementing condition for the stationary problem means that no nontrivial solutions of this kind exist, for the time-dependent problem the appropriate condition is again the exponential decay of such solutions.

Two types of boundary conditions for the Navier-Stokes equations are particularly important. The most commonly used are Dirichlet conditions, i.e. $u = 0$ on the boundary. For these conditions the complementing conditions are trivial, since the energy estimates (6) and (7) carry over without any change to the half-space problem. For problems with free surfaces, traction boundary conditions are important. The discussion of these conditions is much more complicated, and we shall only consider the two-dimensional case.

On the half-plane $x \geq 0$ of R^2, we consider equation (5) with the linearized traction boundary conditions

$$\eta\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) + A_{12}p = 0,$$

$$2\eta\frac{\partial u_1}{\partial x} + (A_{11} - 1)p = 0. \tag{9}$$

We use Fourier transform in the y-direction and Laplace transform in time and look for solutions proportional to $e^{\sigma t}e^{i\alpha y}$. We then have the following set of equations

$$\rho\sigma u_1 = \eta(u_1'' - \alpha^2 u_1) + (A_{11} - 1)p' + A_{12}i\alpha p,$$

$$\rho\sigma u_2 = \eta(u_2'' - \alpha^2 u_2) + A_{12}p' + (A_{22} - 1)i\alpha p, \tag{10}$$

$$u_1' + i\alpha u_2 = 0.$$

The boundary conditions at $x = 0$ read

$$\eta(i\alpha u_1 + u_2') + A_{12}p = 0, \tag{11}$$

$$2\eta u_1' + (A_{11} - 1)p = 0.$$

Here $'$ denotes differentiation with respect to x. The complementing condition says that for Re $\sigma \geq 0$ and $\alpha \neq 0$ there are no nontrivial solutions which decay as $x \to \infty$.

Without loss of generality, we can assume that $\eta = \rho = \alpha = 1$ (this can always be achieved by rescaling). Let us also recall that $A_{11} + A_{22} = 0$, and that ellipticity implies that $A_{12}^2 < (1 - A_{11})(1 - A_{22})$. In particular this means that A_{11} and A_{22} lie between -1 and $+1$. We differentiate the first equation of (10) with respect to x, multiply the second by $i\alpha$, and add them. Using the divergence condition, we obtain

$$(A_{11} - 1)p'' + 2A_{12}ip' - (A_{22} - 1)p = 0. \tag{12}$$

The solutions to this equation are $e^{-\lambda x}$, where

$$\lambda = \frac{-A_{12}i \pm \sqrt{(A_{11}-1)(A_{22}-1) - A_{12}^2}}{1 - A_{11}}. \tag{13}$$

Since we are interested in solutions that decay for $x \to \infty$, we must choose the plus sign in the numerator of (13).

It is easy to see that $p = 0$ does not lead to any nontrivial solutions of (10),(11), hence we set $p = e^{-\lambda x}$ with λ given by (13). From (10) we find

$$u_1 = c_1 e^{-\sqrt{1+\sigma}x} + \frac{(A_{11}-1)\lambda - A_{12}i}{\lambda^2 - 1 - \sigma} e^{-\lambda x}, \tag{14}$$

$$u_2 = c_2 e^{-\sqrt{1+\sigma}x} + \frac{(1-A_{22})i + A_{12}\lambda}{\lambda^2 - 1 - \sigma} e^{-\lambda x}.$$

We have assumed here that $\lambda^2 \neq 1+\sigma$. The case $\lambda^2 = 1+\sigma$ requires a separate discussion, which will be given below. From the incompressibility and boundary conditions, we find three constraints for the two parameters c_1 and c_2. This leads to an eigenvalue problem for σ. The three conditions are

$$-\sqrt{1+\sigma}c_1 + ic_2 = 0,$$

$$ic_1 - \sqrt{1+\sigma}c_2 + i\frac{(A_{11}-1)\lambda - A_{12}i}{\lambda^2 - 1 - \sigma} - \lambda\frac{(1-A_{22})i + A_{12}\lambda}{\lambda^2 - 1 - \sigma} + A_{12} = 0, \tag{15}$$

$$-2\sqrt{1+\sigma}c_1 - 2\lambda\frac{(A_{11}-1)\lambda - A_{12}i}{\lambda^2 - 1 - \sigma} + (A_{11}-1) = 0.$$

Let now $\gamma = \sqrt{1+\sigma}$. After some algebra equations (15) lead to the following equation of third degree for γ

$$(1+\gamma^2)(\lambda+\gamma)(1-A_{11}) - 2iA_{12}(1-\gamma^2) - 2\lambda(1-A_{11}) + 2\gamma(1-A_{22}) = 0. \tag{16}$$

The roots of (16) were computed at a grid of points in the set $\{(A_{11}, A_{12})| -1 < A_{11} < 1, A_{12}^2 < (1 - A_{11}^2)\}$. No roots such that $\mathrm{Re}\,\gamma > 0$ and $\mathrm{Re}\,\sigma = \mathrm{Re}\,(\gamma^2 - 1) \geq 0$ were found.

If we assume $\lambda = \sqrt{1+\sigma}$, we arrive, after some algebra, at the equation

$$i(\lambda^3 + \lambda)(1 - A_{11}) + A_{12}(1 - \lambda^2) + (A_{11} - A_{22})i\lambda = 0. \tag{17}$$

By inserting (13) into this, and comparing the real and imaginary parts, we find that this is the case only if

$$A_{12} = 0, \text{ and } A_{22} = 1. \tag{18}$$

This leads to

$$\lambda = 0, \ \sigma = -1, \tag{19}$$

and ellipticity does not hold for these parameters. We conclude that, at least in two dimensions, the complementing condition is satisfied as long as ellipticity holds.

3. The initial value problem for Dirichlet type boundary conditions

We consider equation (4) on a bounded domain $\Omega \subset R^3$ with smooth boundary. We want to show the existence of solutions satisfying a given initial condition

$$u(x,0) = v(x), \tag{20}$$

and Dirichlet boundary conditions

$$u(x,t) = w(x,t) \text{ on } \partial\Omega. \tag{21}$$

It is assumed that (20) and (21) are compatible with the incompressibility condition and with each other. In addition, we will have to prescribe the average pressure as a function of time,

$$\int_\Omega p(x,t)\, dx = \gamma(t). \tag{22}$$

The latter condition is needed to remove an indeterminacy already present in the ordinary Navier-Stokes equations. There one can add an arbitrary constant to the pressure without changing the flow. If we want to make the pressure uniquely determined, a condition such as (22) is needed. In the present situation, the choice of $\gamma(t)$ will affect the velocity as well as the pressure.

One of the crucial steps in dealing with the Navier-Stokes equations is the elimination of the pressure. In the classical case this is achieved by the Hodge projection or, in other words, by solving a Neumann boundary value problem (see e.g. [9]). In the more complicated situation of equation (4), this Neumann problem is replaced by a nonlinear elliptic equation. We obtain this equation by taking the divergence of (4) and we obtain the associated boundary condition by multiplying (4) by the outer unit normal of the domain. This leads to the problem

$$\operatorname{div}\left\{\eta(p)\Delta u + (2\eta'(p)\mathbf{D} - \mathbf{1})\nabla p\right\} = \operatorname{div}\left\{\rho(u\cdot\nabla)u - f\right\}, \tag{23}$$

with the boundary condition

$$n\cdot\left\{\eta(p)\Delta u + (2\eta'(p)\mathbf{D} - \mathbf{1})\nabla p\right\} = n\cdot\left\{\rho(\dot{w} + (u\cdot\nabla)u) - f\right\}. \tag{24}$$

We want to solve (23), (24) for p when u is given. We require that p satisfy the additional constraint

$$\int_\Omega p(x)\, dx = \gamma, \tag{25}$$

where γ is a given real number. We shall prove the following existence and uniqueness result.

Theorem 1:

Let $0 < \alpha < 1$. Assume that u is in $H^4(\Omega) \cap C^{3,\alpha}(\bar{\Omega})$, f is in $H^2(\Omega) \cap C^{1,\alpha}(\bar{\Omega})$, $\dot{w}$ satisfies the condition $\int_{\partial\Omega} w \cdot n \, dS = 0$ and can be extended to a divergence-free vectorfield, denoted again by $\dot{w}$, which lies in $H^2(\Omega) \cap C^{1,\alpha}(\bar{\Omega})$. Moreover, assume that η is a smooth function of p such that $\eta'(p)$ is bounded and $\lim_{p \to \pm\infty} \frac{\eta(p)}{p} = 0$. Assume, moreover, that the eigenvalues of $\mathbf{D}$ are strictly less than $1/(2 \max \eta')$, if $\max \eta'$ is positive, and strictly greater than $1/(2 \min \eta')$, if $\min \eta'$ is negative. Then (23)-(25) has a unique solution $p \in H^3(\Omega) \cap C^{2,\alpha}(\bar{\Omega})$.

Proof:

We set $g = \rho(u \cdot \nabla)u + \rho\dot{w} - f$. To find solutions of (23)-(25), we look at the family of problems

$$\operatorname{div} \left\{ \tau\eta(p)\Delta u + (2\tau\eta'(p)\mathbf{D} - \mathbf{1})\nabla p \right\} = \operatorname{div} g, \tag{26}$$

with boundary condition

$$n \cdot \left\{ \tau\eta(p)\Delta u + (2\tau\eta'(p)\mathbf{D} - \mathbf{1})\nabla p \right\} = n \cdot g, \tag{27}$$

and the constraint (25).

For $\tau = 0$, we have the ordinary Neumann problem, and we have a unique solution $p \in H^3(\Omega) \cap C^{2,\alpha}(\bar{\Omega})$ for every given $g \in H^2(\Omega) \cap C^{1,\alpha}(\bar{\Omega})$ and $\gamma \in R$. As τ is increased, this solution can be uniquely continued to a family of solutions p_τ, unless one of the following happens.

1. Ellipticity is lost.
2. The norm of p_τ in $H^3 \cap C^{2,\alpha}$ becomes unbounded as τ approaches some value τ_C.
3. The linearized problem has nontrivial solutions.

If none of these three happens up to $\tau = 1$, a solution of (23)-(25) exists. Conversely, if every solution for $\tau = 1$ can be continued back to $\tau = 0$, we have a uniqueness result.

Ellipticity means that the matrix $2\tau\eta'(p)\mathbf{D} - \mathbf{1}$ remains negative definite. According to our assumption on the eigenvalues of $\mathbf{D}$, ellipticity holds up to $\tau = 1$.

The estimates of Agmon, Douglis and Nirenberg can be used to show that the norm of p in $H^3 \cap C^{2,\alpha}$ remains bounded as long as the C^1-norm remains bounded and the ellipticity constant remains strictly positive. We can then use theorem 2.1 on page 476 of [4] to obtain a bound on the C^1-norm provided there is a bound on the maximum norm. If, moreover, we assume that η' is bounded, then lemma 3.1 of [5] applies, and we can replace the maximum norm by the L^1-norm. We can derive an a priori bound on the L^1-norm (in fact on a stronger norm) of p as follows. We multiply (25) by p and integrate over the domain Ω. After an integration by parts, we have

$$\int_\Omega \nabla p(\mathbf{1} - 2\tau\eta'(p)\mathbf{D})\nabla p - \tau\Delta u\eta(p)\nabla p \, dx = \int_\Omega p \operatorname{div} g \, dx - \int_{\partial\Omega} pg \cdot n \, dS. \tag{28}$$

According to our assumption on η, we can, for any $\epsilon > 0$, find a constant $C(\epsilon)$ such that $|\eta(p)| \leq C(\epsilon) + \epsilon|p|$. When we use this estimate in the second term on the left of (28), we get an estimate on the H^1-norm of p.

Finally, we have to show that the linearized problem can have only the trivial solution, i.e. that there can not be nontrivial solutions ϕ of

$$\operatorname{div}\left\{\tau\eta'(p_\tau)\Delta u\phi + 2\tau\eta''(p_\tau)\mathbf{D}\nabla p_\tau\phi + (2\tau\eta'(p_\tau)\mathbf{D} - 1)\nabla\phi\right\} = 0, \tag{29}$$

with boundary condition

$$n \cdot \left\{\tau\eta'(p_\tau)\Delta u\phi + 2\tau\eta''(p_\tau)\mathbf{D}\nabla p_\tau\phi + (2\tau\eta'(p_\tau)\mathbf{D} - 1)\nabla\phi\right\} = 0, \tag{30}$$

and

$$\int_\Omega \phi\, dx = 0. \tag{31}$$

This problem is of the form

$$\operatorname{div}\left\{\mathbf{A}\nabla\phi + f\phi\right\} = 0, \tag{32}$$

with boundary condition

$$n \cdot \left\{\mathbf{A}\nabla\phi + f\phi\right\} = 0, \tag{33}$$

where $\mathbf{A}$ is a positive definite symmetric matrix. Let us consider the parabolic equation

$$\phi_t = \operatorname{div}\left\{\mathbf{A}\nabla\phi + f\phi\right\} \tag{34}$$

with boundary condition (33) and let T be the time 1 evolution operator, i.e. the operator that takes an initial datum $\phi(t = 0)$ to $\phi(t = 1)$. It is easy to verify that $\int_\Omega T\phi\, dx = \int_\Omega \phi\, dx$. It follows from the strong maximum principle for parabolic equations (see chapter 9 of [6]) that $T\phi > 0$ if $\phi \geq 0$ and ϕ does not vanish identically. This implies that, if $T\phi = \phi$, then ϕ must be either positive or negative. Assume the contrary and let ϕ_+ be the positive part of ϕ. If ϕ is not positive or negative, then ϕ_+ is not identical to ϕ and not identical to 0. It follows that $T\phi_+ > T\phi = \phi$ and $T\phi_+ > 0$, hence $T\phi_+ > \phi_+$. But this is a contradiction since $T\phi_+$ must have the same integral as ϕ_+. Hence the equation $T\phi = \phi$ has at most one linearly independent solution. Since T is a compact operator in L^2, the operator $1 - T$ has index zero, and since it maps to functions with zero integral, it is obviously not surjective. Hence the nullspace of T is one-dimensional. The solutions to (32), (33) satisfy $T\phi = \phi$, and can therefore not have zero integral. This completes the proof.

∎

Unfortunately, the assumptions on the function $\eta(p)$ in theorem 1 are not very realistic from a physical point of view. The experiments [3] show that η' increases with p. Hence it may not be realistic to assume that η' is bounded, and it is certainly not realistic that $\eta(p)/p$ tends to zero as $p \to +\infty$. Negative pressures lead to cavitation in reality, and hence any assumptions on $\eta(p)$ for $p \to -\infty$ are academic. Under realistic assumptions, we can therefore not guarantee the existence of a pressure for a prescribed velocity field and body force. Of course an implicit function argument can be used if the velocity and body force are small.

We now turn to the discussion of the Dirichlet initial value problem given by (3) and (20)-(22). We make the following assumptions on the data of the problem. All these assumptions are assumed to hold on some time interval $[0, t_0]$.

(i) The body force f is a uniformly Hölder continuous function of time taking values in $H^2(\Omega)$. Its time derivative is uniformly Hölder continuous with values in $L^2(\Omega)$. Moreover, at $t=0$, f lies in $C^{1,\alpha}(\bar{\Omega})$.

(ii) The function w satisfies the condition $\int_{\partial\Omega} w\cdot n\; dS=0$. It can hence be extended to a divergence-free vector field in the interior of Ω, which we denote again by w. We assume that w is uniformly Hölder continuous with values in $H^4(\Omega)$, its time derivative is uniformly Hölder continuous with values in $H^2(\Omega)$ and its second time derivative is uniformly Hölder continuous with values in $L^2(\Omega)$. At $t=0$, w lies in $C^{3,\alpha}(\bar{\Omega})$. Moreover, $\dot{w}$ lies in $C^{1,\alpha}(\bar{\Omega})$ for $t=0$.

(iii) v is divergence free and lies in $H^4(\Omega)\cap C^{3,\alpha}(\bar{\Omega})$.

(iv) The function γ has a uniformly Hölder continuous derivative.

(v) For $t=0$, equations (23)-(25) (with $u=v$) have a unique solution $p_0\in H^3(\Omega)\cap C^{2,\alpha}(\bar{\Omega})$ (e.g. if the assumptions of theorem 1 hold). The matrix $2\eta'(p_0)[\nabla v+(\nabla v)^T]-\mathbf{1}$ is strictly negative definite on all of $\bar{\Omega}$.

(vi) $v(x)=w(x,0)$ on $\partial\Omega$, and $\dot{u}(x,t=0)$ as computed from (3) agrees with $\dot{w}(x,0)$ on $\partial\Omega$.

Theorem 2:

Assume that assumptions (i)-(vi) hold. Then equation (3) with the data (20)-(22) has a unique solution on some time interval $[0,T_1)$ *with* $0<T_1\le t_0$. *The solution is such that* $u\in C^0([0,T_1),H^4(\Omega))\cap C^1([0,T_1),H^2(\Omega))\cap C^2([0,T_1),L^2(\Omega))$ *and* $p\in C^0([0,T_1),H^3(\Omega))\cap C^1([0,T_1),H^1(\Omega))$.

Proof:

We shall look not only at equation (3) itself, but also at the time differentiated equation. Let us denote $a=\dot{u}-\lambda u$ (λ is a constant to be specified later) and $q=\dot{p}$. On differentiating (3), we obtain the following evolution problem for a

$$\rho[\dot{a}+\lambda a+\lambda^2 u+((a+\lambda u)\cdot\nabla)u+(u\cdot\nabla)(a+\lambda u)]=$$

$$\operatorname{div}\left[\eta'(p)q(\nabla u+(\nabla u)^T)+\eta(p)(\nabla(a+\lambda u)+(\nabla(a+\lambda u))^T)\right]-\nabla q+\dot{f},\tag{35}$$

$$\operatorname{div} a=0.$$

Moreover, we have the boundary condition

$$a(x,t)=\dot{w}(x,t)-\lambda w(x,t)\text{ on }\partial\Omega,\tag{36}$$

and the average of q is given by

$$\int_\Omega q(x,t)\;dx=\dot{\gamma}(t).\tag{37}$$

Finally, an initial condition for a is obtained from equation (3)

$$a(x,0)=-\lambda v-(v\cdot\nabla)v+\frac{1}{\rho}\left\{\operatorname{div}\left[\eta(p_0)(\nabla v+(\nabla v)^T\right]-\nabla p_0+f(x,0)\right\}.\tag{38}$$

Here p_0 is the initial pressure which is guaranteed to exist by assumption (v).

The strategy is now to express u and p in terms of a using equations (3), (21) and (22). By inserting the result into (35), we get a problem involving only a and q. From this, we eliminate q by solving a Neumann problem as above. This finally yields an evolution equation for a, which we shall treat using the theory of analytic semigroups.

At time $t = 0$, we have (3), (21) and (22) satisfied with $u = v$, $p = p_0$ and $a = a(x,0)$ as given by (38). In the neighborhood of these data, we want to show that if $\lambda \in R$ is chosen large enough, then we can uniquely resolve the equations for $u \in H^4$ and $p \in H^3$ in terms of $a \in H^2$, $f \in H^2$, $w \in H^4$ and γ. In a weaker norm, we shall get $u \in H^3$ and $p \in H^2$ in terms of $a \in H^1$, $f \in H^1$, $w \in H^3$, and γ. In order to show this, we use the implicit function theorem. For this, we have to look at the linearized equations. Linearizing (3) with respect to u and p, we arrive at the following problem

$$\operatorname{div}\left[\eta(p_0)(\nabla u+(\nabla u)^T)+\eta'(p_0)p(\nabla v+(\nabla v)^T)\right]-\nabla p-\rho((v\cdot\nabla)u+(u\cdot\nabla)v)-\rho\lambda u = g, \tag{39}$$

$$\operatorname{div} u = 0.$$

We have to show that (39) together with the inhomogenous Dirichlet conditions

$$u = w \text{ on } \partial\Omega, \tag{40}$$

and

$$\int_\Omega p\, dx = \gamma, \tag{41}$$

has a unique solution $u \in H^4$, $p \in H^3$ for every $g \in H^2$, every w that is the trace of a divergence-free vectorfield in H^4, and every $\gamma \in R$ (plus the corresponding assertion for the weaker norms). The inhomogeneous boundary terms can be absorbed into g by letting $u = w + \tilde{u}$, where w is an extension of the boundary data, so we can henceforth assume that $w = 0$, and also that $\gamma = 0$.

We shall derive estimates that imply uniqueness of solutions to (39)-(41) for large enough λ. Moreover, these estimates are such that they hold uniformly for a family of problems which continuously connect (39) to the standard Navier-Stokes equations (we can use a homotopy that takes η to a constant function). The results of chapter 12 in [1] then imply existence.

Hence we now let $g = 0$ in (39), $w = 0$ in (40), $\gamma = 0$ in (41) and we assume that λ is positive and large. We first multiply (39) by $\{1 - \eta'(p_0)(\nabla v + (\nabla v)^T)\}^{-1}u$ and integrate over the domain Ω. After an integration by parts this yields an estimate of the form (provided λ is large enough)

$$\|u\|_{H^1} + \lambda\|u\|_{L^2} \leq C_1\|p\|_{L^2}. \tag{42}$$

The elliptic estimates of Agmon, Douglis and Nirenberg [1] lead to

$$\|u\|_{H^2} + \|p\|_{H^1} \leq C_2(\lambda\|u\|_{L^2} + \|p\|_{L^2}). \tag{43}$$

By taking the two together, we arrive at

$$\|u\|_{H^2} + \|p\|_{H^1} + \lambda\|u\|_{L^2} \leq C_3\|p\|_{L^2}. \tag{44}$$

We can use the same procedure as in the proof of theorem 1 to eliminate the pressure, and we note that the equation for the pressure is independent of λ. This gives p as a linear function of u and a straightforward energy estimate leads to

$$\|p\|_{H^1} \leq C_4\|u\|_{H^2} \tag{45}$$

holds. Hence the mapping $u \in H^2 \to p \in L^2$ is compact. It follows that for every $\epsilon > 0$ there is a constant $C(\epsilon)$ such that

$$\|p\|_{L^2} \leq \epsilon\|u\|_{H^2} + C(\epsilon)\|u\|_{L^2} \tag{46}$$

(use approximation by operators of finite rank and the fact that L^2 is dense in the dual of H^2). We now obtain the desired uniqueness result by inserting (46) into (44) and choosing ϵ and λ in such a way that $\epsilon C_3 < 1$ and $C(\epsilon)C_3 < \lambda$.

Using the same steps, we can obtain a resolvent estimate for complex λ. Let λ be in any sector Σ of the complex plane which excludes the negative half-axis and let $|\lambda|$ be sufficiently large. Moreover, assume that $w = 0$, $\gamma = 0$. Then the solution to (39)-(41) satisfies an estimate of the form

$$\|u\|_{H^2} + \|p\|_{H^1} + |\lambda|\|u\|_{L^2} \leq C\|g\|_{L^2} \tag{47}$$

with a constant C that does not depend on λ.

We now turn to equations (35)-(37). According to the above we may regard u and p as having been expressed in terms of a and the data of the problem. Moreover, we can subtract appropriate reference functions from a and q such that (36) and (37) are transformed to a homogeneous form. Finally, we can eliminate q by using the procedure in the proof of theorem 1. In this way, we obtain an evolution problem involving only a as an unknown. To this problem we can now apply the theory of Sobolevskii [7]. The essential assumption of this theory is that the terms of highest differential order generate an analytic semigroup. The terms of highest order are those terms on the right hand side of (35) which involve second derivatives of a and first derivatives of q. These are of precisely the same form as in (39), and hence the needed resolvent estimate follows from (47). The remaining assumptions of [7] concerning smoothness of the nonlinearities and regularity of the data are easily checked. This completes the proof.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 2886	2. GOVT ACCESSION NO. AD-A163726	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) SOME REMARKS ON THE NAVIER-STOKES EQUATIONS WITH A PRESSURE-DEPENDENT VISCOSITY		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Michael Renardy		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041 MCS-8215064; DMS-8451761
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53705		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below.		12. REPORT DATE November 1985
		13. NUMBER OF PAGES 12
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Navier-Stokes equations
pressure dependent viscosity
nonlinear Neumann problems
complementing condition
initial value problems
change of type

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We discuss the Navier-Stokes equations for an incompressible fluid with a viscosity that is allowed to depend on the pressure. Ellipticity and the complementing condition of Agmon, Douglis and Nirenberg [1] are discussed. It is found that the pressure dependence of viscosity leads to the possibility of a change of type. It is shown that the Dirichlet initial-boundary value problem is well posed as long as the equations do not change type.

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